Pergamon

Computers Math. Applic. Vol. 28, No. 9, pp. 89–99, 1994

0898-1221/94 $7.00 + 0.00

0898-1221(94)00181-2

The Stability Properties of Nonlinear Measure Large Scale Systems with Impulse Effect

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(Received June 1993; accepted July 1993)

Abstract—In this paper, we studied the stability properties of nonlinear measure large scale systems by means of the lumped Gauss-Seidel iteration method, which avoided the difficulties of constructing Lyapunov functions. The explicit algebraic criteria of the stability, uniform stability, asymptotic stability, uniform asymptotic stability, and exponential stability for nonlinear measure large scale systems are established.

Keywords—Measure differential large scale systems, Distributional derivative, Impulsive solutions, Lumped Gauss-Seidel iteration, Stability.

1. INTRODUCTION

Many evolution processes, such as biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, frequency modulated systems and motion of missiles or airplanes are characterized by the fact that, at certain moments of time, they experience a change of state abruptly. These perturbations act instantaneously, that is, in the form of an impulse. Thus, it is natural and necessary to study systems involving impulses which can be successfully described by the measure differential systems [1–3].

In this paper, we shall consider the following nonlinear measure large scale system

$$D\mathbf{y}_i = \mathbf{A}_i(t)\,\mathbf{y}_i\,Du_i + \sum_{j=1}^{r} \mathbf{F}_{ij}(t, \mathbf{y}_j(t))\,Du_j + \mathbf{G}_i(t, \mathbf{Y}(t))\,Du_i, \qquad i = 1, \dots, r, \tag{1.1}$$

where Du_i denote the distribution derivative of the function u_i. If u_i are functions of bounded variation, Du_i can be identified with the Stieltjes measure, which has the effect of suddenly changing the state of the system at the points of discontinuity of u_i. The fact that the solutions of (1.1) are discontinuous (i.e., functions of bounded variation) offers many difficulties in applying the usual techniques of large scale system theory. In view of this, there is little research on large scale measure differential systems except for references [4,5].

The object of this paper is to investigate some stability properties of solutions of the measure large scale system (1.1), by means of lumped iteration [6], which is different from the methods in [1–3, 7–9]. Section 2 is devoted to preliminaries and a basic lemma. Section 3 contains main theorems on stability, uniform asymptotic stability, and exponential stability of the zero solution of (1.1).

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2. PRELIMINARIES AND BASIC LEMMA

Consider the measure differential large scale system

$$D\mathbf{y}_i = \mathbf{A}_i(t)\,\mathbf{y}_i\,Du_i + \sum_{j=1}^{r} \mathbf{F}_{ij}(t, \mathbf{y}_j(t))\,Du_j + \mathbf{G}_i(t, \mathbf{Y}(t))\,Du_i, \qquad i = 1, \dots, r, \tag{2.1}$$

where the symbol D stands for the derivative in the sense of distributions, $\mathbf{y}_i \in \mathbb{R}^{n_i}$, $\mathbf{A}_i(t)$ are $n_i \times n_i$ continuous matrices on $J = [t_0, \infty)$, $t_0 \geq 0$; $u_i : J \to \mathbb{R}$ are functions of bounded variation, right-continuous on every compact subinterval of J; $\mathbf{F}_{ij}(t, \mathbf{y}_j(t)) : J \times \mathbb{R}^{n_j} \to \mathbb{R}^{n_i}$, $\mathbf{G}_i(t, \mathbf{Y}(t)) : J \times \mathbb{R}^n \to \mathbb{R}^{n_i}$ are integrable (in the sense of Lebesgue-Stieltjes) with respect to u_j and u_i, respectively; and $\mathbf{F}_{ij}(t, \mathbf{0}) \equiv 0$, $\mathbf{G}_i(t, \mathbf{0}) \equiv 0$, $i, j = 1, \dots, r$, $\sum_{j=1}^{r} n_j = n$.

For the existence and uniqueness of solutions of system (2.1), refer to [7]. In the sequel, we shall assume that the solutions of system (2.1) exist and are unique for $t \geq t_0$.

In general, a function of bounded variation and right-continuous contains two parts: one is an absolutely continuous function and the other is a singular function. Without loss of generality, we assume that

$$u_i(t) = t + \sum_{k=1}^{\infty} \mathbf{a}_{ik}\,H_k(t), \qquad i = 1, \dots, r, \tag{2.2}$$

where $t_1 < t_2 < \cdots < t_k < \cdots$, and $t_k \to \infty$ as $k \to \infty$, are discontinuity points, $t_1 > t_0$; $\mathbf{a}_{ik}$ are constants, and

$$H_k(t) = \begin{cases} 0, & t < t_k, \\ 1, & t \geq t_k. \end{cases}$$

It is easy to see that $Du_i = 1 + \sum_{k=1}^{\infty} \mathbf{a}_{ik}\,\delta(t_k)$, where $\delta(t_k)$ is a Dirac measure condensed at t_k, $u_i' = 1$ almost everywhere on J, and for any $t \in J$, there exists an unique $k \in N$ such that $t \in [t_{k-1}, t_k)$.

The isolated subsystems of (2.1) are such that

$$D\mathbf{y}_i = \mathbf{A}_i(t)\,\mathbf{y}_i\,Du_i, \qquad i = 1, \dots, r. \tag{2.3}$$

Corresponding to (2.3), the ordinary differential subsystems are such that

$$\mathbf{y}_i' = \mathbf{A}_i(t)\,\mathbf{y}_i, \qquad i = 1, \dots, r. \tag{2.4}$$

Let $\mathbf{P}(t, t_0) = \operatorname{diag}(P_1(t, t_0), \dots, P_r(t, t_0))$ denote the Cauchy matrix relative to the system (2.4). Letting

$$\mathbf{B}_{ik} = \mathbf{E} - \mathbf{a}_{ik}\,\mathbf{A}_i(t_k), \quad i = 1, \dots, r, \quad k = 1, 2, \dots, \tag{2.5}$$

where $\mathbf{E}$ is the $n_i \times n_i$ identity matrix, we state the following lemma.

LEMMA 2.1. *Let matrix $\mathbf{B}_{ik}$ be not singular for every integer $k \geq 1$ and $i = 1, \dots, r$. Then, for $t \in [t_{k-1}, t_k)$ the nontrivial solution $\mathbf{Y}(t) = \mathbf{Y}(t, t_0, \mathbf{Y}_0) = \operatorname{Col}(\mathbf{y}_1(t), \dots, \mathbf{y}_r(t))$ of (2.1) is given by*

$$\begin{aligned}
\mathbf{Y}_i(t) &= P_i(t, t_{k-1}) \left[\prod_{j=1}^{k-1} \mathbf{B}_{i,k-j}^{-1}\, P_i(t_{k-j}, t_{k-j-1}) \right] \mathbf{y}_{i0} \\
&\quad + \int_{t_0}^{t} P_i(t, s) \left[\sum_{j=1}^{r} \mathbf{F}_{ij}(s, \mathbf{y}_j(s))\,du_j(s) + \mathbf{G}_i(s, \mathbf{Y}(s))\,du_i(s) \right] \\
&\quad + P_i(t, t_{k-1}) \sum_{s=1}^{k-1} \mathbf{a}_{is} \left[\prod_{j=1}^{k-1-s} \mathbf{B}_{i,k-j}^{-1}\, P_i(t_{k-j}, t_{k-j-1}) \right] \mathbf{B}_{is}^{-1}\, \mathbf{A}_i(t_s)
\end{aligned}$$

$$\times \int_{t_0}^{t_s} P_i(t_s,\xi) \left[\sum_{j=1}^{r} \mathbf{F}_{ij}(\xi, y_j(\xi))\, du_j(\xi) + \mathbf{G}_i(\xi, \mathbf{Y}(\xi))\, du_i(\xi) \right], \tag{2.6}$$

$i = 1, \dots, r$, *where if* $k = 1$, *then* $\prod_{j=1}^{k-1}$ *becomes the identity matrix* $\mathbf{E}$.

Lemma 2.1 can be regarded as the variation of parameters formula of nonlinear measure large scale systems. The proof of Lemma 2.1 is similar to our earlier work [4,5]. The details are omitted.

3. STABILITY THEOREM

In this section, $A(\mathbf{a}_{ij}) \geq B(\mathbf{b}_{ij})$ implies that $\mathbf{a}_{ij} \geq \mathbf{b}_{ij}$ $(i,j = 1,\dots,n)$, and $\mathrm{Col}(x_1,\dots,x_n) \geq \mathrm{Col}(y_1,\dots,y_n)$ implies that $x_i \geq y_i$ $(i = 1,\dots,n)$. We always assume that $\mathbf{B}_{ik} = \mathbf{E} - \mathbf{a}_{ik}\,\mathbf{A}_i(t_k)$ is not singular for any integer $k \geq 1$ and $i = 1,\dots,r$.

We need following hypothesis in our subsequent discussion. For $i,j = 1,\dots,r$ and $t \geq t_0$:

(H$_1$) $\mathbf{F}_{ij}(t,\mathbf{y}_j)$ and $\mathbf{G}_i(t,\mathbf{Y})$ are continuous with respect to $\mathbf{y}_j$ and $\mathbf{Y}$, respectively, for fixed $t \in J$, and there exists a constant $H > 0$ and scalar functions $f_{ij}(t)$, $g_{ij}(t)$, which are integrable with respect to the Lebesgue-Stieltjes measures $d|u_j|$ $\left(d|u_j|(t) = d\left(\overset{t}{\underset{t_0}{V}}(u_j)\right)\right)$ and $d|u_i|$, respectively, such that

$$\|\mathbf{G}_i(t,\mathbf{X}) - \mathbf{G}_i(t,\mathbf{Y})\| \leq \sum_{j=1}^{r} \mathbf{g}_{ij}(t)\, \|\mathbf{x}_j - \mathbf{y}_j\|,$$

$$\|\mathbf{F}_{ij}(t,\mathbf{x}_j) - \mathbf{F}_{ij}(t,\mathbf{y}_j)\| \leq \mathbf{f}_{ij}(t)\, \|\mathbf{x}_j - \mathbf{y}_j\|, \qquad \forall \mathbf{X},\mathbf{Y} \in \mathbb{R}^n, \quad \|\mathbf{X}\| < H, \quad \|\mathbf{Y}\| < H.$$

(H$_2$) $\|P_i(t,t_0)\| \leq M_i \exp\left(-\int_{t_0}^t \lambda_i(\xi)\,d\xi\right) \leq M_i \exp\left(-\int_{t_0}^t \lambda(\xi)\,d\xi\right)$, where M_i are constants and $\lambda(t)$, $\lambda_i(t) \in C(J)$, $i = 1,\dots,r$.

(H$_3$) $\int_{t_0}^t (M_i + c_i\,\alpha_i) \exp\left(-\int_s^t (\lambda_i(\xi) - \lambda(\xi))\,d\xi\right) [f_{ij}(s)\,d|u_j|(s) + g_{ij}(s)\,d|u_i|(s)] \leq b_{ij} = \text{const.}$, where $\alpha_i = \max_{k\in\mathbb{N}} \|\mathbf{a}_{ik}\,\mathbf{A}_i(t_k)\|$, $c_i = M_i^*\,\beta_i$, for any $k \in \mathbb{N}$, $M_i^K \leq M_i^*$, $\prod_{j=1}^k \|\mathbf{B}_{ij}^{-1}\| \leq \beta_i$. We define

$$\mathbf{D} = (\mathbf{E} - \mathbf{B}_1)^{-1}\mathbf{B}_2,$$

$$\mathbf{B}_1 = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ b_{21} & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{r,r1} & 0 \end{bmatrix},$$

$$\mathbf{B}_2 = \mathbf{B} - \mathbf{B}_1, \qquad \mathbf{B} = (\mathbf{b}_{ij})_{r\times r};$$

$\rho(\mathbf{D})$ is the spectral radius of matrix $\mathbf{D} = (\mathbf{d}_{ij})_{r\times r}$.

THEOREM 3.1. *Let hypotheses* (H$_1$)–(H$_3$) *hold. If* $\rho(\mathbf{D}) < 1$, *then we have the conclusions:*

(I) *For the following conditions:*

(1) $\int_{t_0}^t \lambda(s)\,ds \geq T(t_0) = \text{const.}$, $t \geq t_0$;

(2) $\int_{t_0}^t \lambda(s)\,ds \geq T = \text{const.}$, $t \geq t_0$;

(3) $\int_{t_0}^t \lambda(s)\,ds = +\infty$;

(4) $\int_{t_0}^t \lambda(s)\,ds \to +\infty$ *(uniformly in respect to* t_0*)*, $(t - t_0 \to +\infty)$;

(5) $\int_{t_0}^t \lambda(s)\,ds \geq \beta\,(t - t_0)$, $\beta = \text{const.} > 0$, $t \geq t_0$,

the trivial solution of (2.1) *is:*

(1) *stable;*
(2) *uniformly stable;*
(3) *asymptotically stable;*
(4) *uniformly asymptotically stable;*
(5) *exponentially stable, respectively.*

(II) *If one of the conditions* (3), (4), *or* (5) *in* (I) *holds, then the attractive region of* (2.1) *is given by*

$$\|\mathbf{Y}_0\| \le H_1 \equiv H / \left[c\, b^{r-1} \left\|(\mathbf{E}-\mathbf{D})^{-1}\right\| \, \|\operatorname{Col}(1,\dots,1)\| \sup_{t\ge t_0} \exp\left(-\int_{t_0}^{t} \lambda(s)\, ds \right) \right], \tag{3.1}$$

where $\mathbf{E}$ *is the* $r\times r$ *identity matrix,* $b = \max_{1\le i\le r}\left\{1+\sum_{j=1}^{i-1} \mathbf{b}_{ij}\right\}$, $c = \max_{1\le i\le r}\{c_i\}$.

PROOF. Let $\mathbf{Y}(t) = \mathbf{Y}(t,t_0,=\mathbf{Y}_0) = \operatorname{Col}(\mathbf{y}_1(t),\dots,\mathbf{y}_r(t))$ be any nontrivial solution of (2.1), $\mathbf{y}_i(t) = \mathbf{y}_i(t,t_0,\mathbf{y}_{i0})$. By the variation of parameters formula (Lemma 2.1), for $t\in[t_{k-1},t_k)$, we get

$$\begin{aligned}
\mathbf{y}_i(t) &= P_i(t,t_{k-1}) \left[\prod_{j=1}^{k-1} \mathbf{B}_{i,k-j}^{-1} P_i(t_{k-j},t_{k-j-1}) \right] \mathbf{y}_{i0} \\
&+ \int_{t_0}^{t} P_i(t,\xi) \left[\sum_{j=1}^{r} \mathbf{F}_{ij}(\xi,\mathbf{y}_j(\xi))\, du_j(\xi) + \mathbf{G}_i(\xi,\mathbf{Y}(\xi))\, du_i(\xi) \right] \\
&+ P_i(t,t_{k-1}) \sum_{s=1}^{k-1} \mathbf{a}_{is} \left[\prod_{j=1}^{k-1-s} \mathbf{B}_{i,k-j}^{-1} P_i(t_{k-j},t_{k-j-1}) \right] \mathbf{B}_{is}^{-1} \mathbf{A}_i(t_s) \\
&\times \int_{t_0}^{t_s} P_i(t_s,\xi) \left[\sum_{j=1}^{r} \mathbf{F}_{ij}(\xi,\mathbf{y}_j(\xi))\, du_j(\xi) + \mathbf{G}_i(\xi,\mathbf{Y}(\xi))\, du_i(\xi) \right].
\end{aligned} \tag{3.2}$$

Make the lumped Gauss-Seidel iteration [6] for Equation (3.2):

$$\begin{aligned}
\mathbf{y}_i^{(m)}(t) &= P_i(t,t_{k-1}) \left[\prod_{j=1}^{k-1} \mathbf{B}_{i,k-j}^{-1} P_i(t_{k-j},t_{k-j-1}) \right] \mathbf{y}_{i0} \\
&+ \int_{t_0}^{t} P_i(t,\xi) \Bigg[\sum_{j=1}^{i-1} \mathbf{F}_{ij}\left(\xi,\mathbf{y}_j^{(m)}(\xi)\right) du_j(\xi) + \sum_{j=i}^{r} \mathbf{F}_{ij}(\xi,\mathbf{y}_j^{(m-1)}(\xi))\, du_j(\xi) \\
&+ \mathbf{G}_i\left(\xi,\mathbf{y}_1^{(m)}(\xi),\dots,\mathbf{y}_{i-1}^{(m)}(\xi),\, \mathbf{y}_i^{(m-1)}(\xi),\dots,\mathbf{y}_r^{(m-1)}(\xi)\right) du_i(\xi) \Bigg] \\
&+ P_i(t,t_{k-1}) \sum_{s=1}^{k-1} \mathbf{a}_{is} \left[\prod_{j=1}^{k-1-s} \mathbf{B}_{i,k-j}^{-1} P_i(t_{k-j},t_{k-j-1}) \right] \mathbf{B}_{is}^{-1} \mathbf{A}_i(t_s) \\
&\times \int_{t_0}^{t_s} P_i(t_s,\xi) \Bigg[\sum_{j=1}^{i-1} \mathbf{F}_{ij}\left(\xi,\mathbf{y}_j^{(m)}(\xi)\right) du_j(\xi) + \sum_{j=i}^{r} \mathbf{F}_{ij}\left(\xi,\mathbf{y}_j^{(m-1)}(\xi)\right) du_j(\xi) \\
&+ \mathbf{G}_i\left(\xi,\mathbf{y}_1^{(m)}(\xi),\dots,\mathbf{y}_{i-1}^{(m)}(\xi),\, \mathbf{y}_i^{(m-1)}(\xi),\dots,\mathbf{y}_r^{(m-1)}(\xi)\right) du_i(\xi) \Bigg];
\end{aligned}$$

$$
\begin{aligned}
\mathbf{y}_i^{(0)}(t) &= P_i(t,t_{k-1}) \left[\prod_{j=1}^{k-1} \mathbf{B}_{i,k-j}^{-1}\, P_i(t_{k-j},t_{k-j-1})\right] \mathbf{y}_{i0} \\
&\quad + \int_{t_0}^{t} P_i(t,s) \left[\sum_{j=1}^{i-1} F_{ij}\left(s,\mathbf{y}_j^{(0)}(s)\right)\, du_j(s)\right. \\
&\quad \left. + G_i\left(s,\mathbf{y}_1^{(0)}(s),\dots,\mathbf{y}_{i-1}^{(0)}(s),0,\dots,0\right)\, du_i(s)\right] \\
&\quad + P_i(t,t_{k-1}) \sum_{s=1}^{k-1} \mathbf{a}_{is} \left[\prod_{j=1}^{k-1-s} \mathbf{B}_{i,k-j}^{-1}\, P_i(t_{k-j},t_{k-j-1})\right] \mathbf{B}_{is}^{-1}\, \mathbf{A}_i(t_s) \\
&\quad \times \int_{t_0}^{t_s} P_i(t_s,\xi) \left[\sum_{j=1}^{i-1} \mathbf{F}_{ij}\left(\xi,\mathbf{y}_j^{(0)}(\xi)\right)\, du_j(\xi)\right. \\
&\quad \left. + \mathbf{G}_i\left(\xi,\mathbf{y}_1^{(0)}(\xi),\dots,\mathbf{y}_{i-1}^{(0)}(\xi),0,\dots,0\right)\, du_i(s)\right] \\
&\quad t \in [t_{k-1},t_k), \quad i=1,\dots,r, \quad m=1,2,\dots.
\end{aligned} \tag{3.3}
$$

In view of the fact that $P_i(t,t_{k-1}) \left[\prod_{j=1}^{k-1} \mathbf{B}_{i,k-j}^{-1}\, P_i(t_{k-j},t_{k-j-1})\right] \mathbf{y}_{i0}$ is the solution of (2.3) [10] on $[t_{k-1},t_k)$, it is easy to see that $\left\{\mathbf{y}_i^{(m)}(t)\right\}$ is right-continuous and bounded variation sequence on $[t_{k-1},t_k)$. First of all, we make following estimates:

$$
\begin{aligned}
&\left\| P_i(t,t_{k-1}) \left[\prod_{j=1}^{k-1} \mathbf{B}_{i,k-j}^{-1}\, P_i(t_{k-j},t_{k-j-1})\right] \mathbf{y}_{i0} \right\| \\
&\qquad \le M_i \left\|\mathbf{y}_{i0}\right\| \exp\left(-\int_{t_{k-1}}^{t} \lambda_i(\xi)\, d\xi\right) \left[\prod_{j=1}^{k-1} \left\|\mathbf{B}_{i,k-j}^{-1}\right\| M_i \exp\left(-\int_{t_{k-j-1}}^{t_{k-j}} \lambda_i(\xi)\, d\xi\right)\right] \\
&\qquad \le M_i^k\, \beta_i \left\|\mathbf{y}_{i0}\right\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right) \\
&\qquad \le c_i \left\|\mathbf{y}_{i0}\right\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right), \qquad i=1,\dots,r;
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
&\left\| P_i(t,t_{k-1}) \sum_{s=1}^{k-1} \mathbf{a}_{is} \left[\prod_{j=1}^{k-1-s} \mathbf{B}_{i,k-j}^{-1}\, P_i(t_{k-j},t_{k-j-1})\right] \mathbf{B}_{is}^{-1}\, \mathbf{A}_i(t_s) \right. \\
&\qquad \left. \times \int_{t_0}^{t_s} P_i(t_s,\xi) \left[\sum_{j=1}^{r} \mathbf{F}_{ij}(\xi,\mathbf{y}_j(\xi))\, du_j(\xi) + \mathbf{G}_i(\xi,\mathbf{Y}(\xi))\, du_i(\xi)\right] \right\| \\
&\quad \le M_i \exp\left(-\int_{t_{k-1}}^{t} \lambda_i(\xi)\, d\xi\right) \sum_{s=1}^{k-1} \left[\prod_{j=s}^{k-1} \|\mathbf{B}_{ij}^{-1}\|\right] M_i^{k-1-s} \exp\left(-\int_{t_s}^{t_{k-1}} \lambda_i(\xi)\, d\xi\right) \\
&\qquad \times \|\mathbf{a}_{is}\, \mathbf{A}_i(t_s)\| \int_{t_0}^{t_s} M_i \exp\left(-\int_{\tau}^{t_s} \lambda_i(\xi)\, d\xi\right) \sum_{j=1}^{r} \|\mathbf{y}_j(\tau)\| \\
&\qquad \times \left[\mathbf{f}_{ij}(\tau)\, d|u_j|(\tau) + \mathbf{g}_{ij}(\tau)\, d|u_i|(\tau)\right] \\
&\quad \le c_i\, \alpha_i \sum_{s=1}^{k-1} \int_{t_0}^{t_s} \exp\left(-\int_{\tau}^{t} \lambda_i(\xi)\, d\xi\right) \sum_{j=1}^{r} \|y_j(\tau)\| \left[f_{ij}(\tau)\, d|u_j|(\tau) + g_{ij}(\tau)\, d|u_i|(\tau)\right]
\end{aligned}
$$

$$\leq c_i\,\alpha_i \int_{t_0}^{t} \exp\left(-\int_s^t \lambda_i(\xi)\,d\xi\right) \sum_{j=1}^{r} \|y_j(s)\|$$
$$\times\, [\mathbf{f}_{ij}(s)\,d|u_j|(s) + \mathbf{g}_{ij}(s)\,d|u_i|(s)]\,, \qquad i = 1,\ldots,r. \tag{3.5}$$

From (3.3)–(3.5), we get

$$\left\|\mathbf{y}_1^{(0)}(t)\right\| \leq c\,\|\mathbf{Y}_0\|\,\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right);$$

$$\left\|\mathbf{y}_2^{(0)}(t)\right\| \leq c\,\|\mathbf{Y}_0\|\,\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right) + \int_{t_0}^{t} (M_2 + c_2\,\alpha_2)\,\exp\left(-\int_s^t \lambda_2(\xi)\,d\xi\right)$$
$$\times \left\|\mathbf{y}_1^{(0)}(s)\right\| [f_{21}(s)\,d|u_1|(s) + g_{21}(s)\,d|u_2|(s)]$$
$$\leq (c\,\|\mathbf{Y}_0\| + b_{21}\,c\,\|\mathbf{Y}_0\|)\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right)$$
$$\leq b\,c\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right);$$

$$\left\|\mathbf{y}_3^{(0)}(t)\right\| \leq c\,\|\mathbf{Y}_0\|\,\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right) + (M_3 + c_3\,\alpha_3)\int_{t_0}^{t} \exp\left(-\int_s^t \lambda_3(\xi)\,d\xi\right)$$
$$\times \Big[f_{31}(s)\left\|\mathbf{y}_1^{(0)}(s)\right\| d|u_1|(s) + f_{32}(s)\left\|\mathbf{y}_2^{(0)}(s)\right\| d|u_2|(s)$$
$$+ \left(g_{31}(s)\left\|\mathbf{y}_1^{(0)}(s)\right\| + g_{32}(s)\left\|\mathbf{y}_2^{(0)}(s)\right\|\right) d|u_3|(s)\Big]$$
$$\leq (c\,\|\mathbf{Y}_0\| + b_{31}\,c\,\|\mathbf{Y}_0\| + b_{32}\,b\,c\,\|\mathbf{Y}_0\|)\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right)$$
$$\leq b^2\,c\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right);$$

$$\cdots \qquad \cdots$$

$$\left\|\mathbf{y}_r^{(0)}(t)\right\| \leq c\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right) + (M_r + c_r\,\alpha_r)\int_{t_0}^{t} \exp\left(-\int_s^t \lambda_r(\xi)\,d\xi\right)$$
$$\times \left(\sum_{j=1}^{r-1} \left\|\mathbf{y}_j^{(0)}(s)\right\| [f_{rj}(s)\,d|u_j|(s) + g_{rj}(s)\,d|u_r|(s)]\right)$$
$$\leq c\,\|\mathbf{Y}_0\|\,\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right) + \sum_{j=1}^{r-1} b_{rj}\,b^{j-1}\,c\|\mathbf{Y}_0\|\,\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right)$$
$$\leq b^{r-1}\,c\,\|\mathbf{Y}_0\|\,\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right).$$

Therefore,

$$\left\|\mathbf{Y}_i^{(0)}(t)\right\| \leq L\,\|\mathbf{Y}_0\|\,\exp\left(-\int_{t_0}^{t} \lambda(\xi)\,d\xi\right), \qquad i = 1,\ldots,r, \tag{3.6}$$

where $L = b^{r-1}\,c$. Consider that

$$\left\|\mathbf{y}_1^{(1)}(t) - \mathbf{y}_1^{(0)}(t)\right\| \leq (M_1 + c_1\,\alpha_1)\int_{t_0}^{t} \exp\left(-\int_s^t \lambda_1(\xi)\,d\xi\right)$$
$$\times \sum_{j=1}^{r} \left\|\mathbf{y}_j^{(0)}(s)\right\| [f_{1j}(s)\,d|u_j|(s) + g_{1j}(s)\,d|u_1|(s)]$$
$$\leq (M_1 + c_1\,\alpha_1)\int_{t_0}^{t} \exp\left(-\int_s^t (\lambda_1(\xi) - \lambda(\xi))\,d\xi\right) L\,\|\mathbf{Y}_0\| \tag{3.7}$$

$$\times \sum_{j=1}^{r} [f_{1j}(s)\, d|u_j|(s) + g_{1j}(s)\, d|u_1|(s)] \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right)$$
$$\le \sum_{j=1}^{r} b_{1j}\, L\, \|\mathbf{Y}_0\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right) \triangleq \phi_1\, L\, \|\mathbf{Y}_0\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right); \quad (3.7, \text{cont.})$$

$$\begin{aligned}
\left\|\mathbf{y}_2^{(1)}(t) - \mathbf{y}_2^{(0)}(t)\right\| &\le (M_2 + c_2\, \alpha_2) \int_{t_0}^{t} \exp\left(-\int_{s}^{t} \lambda_2(\xi)\, d\xi\right) \\
&\quad \times \Bigg[\left\|\mathbf{y}_1^{(1)}(s) - \mathbf{y}_1^{(0)}(s)\right\| (f_{21}(s)\, d|u_1|(s) + g_{21}(s)\, d|u_2|(s)) \\
&\quad + \sum_{j=2}^{r} \left\|\mathbf{y}_j^{(0)}(s)\right\| (f_{2j}(s)\, d|u_j|(s) + g_{2j}(s)\, d|u_2|(s))\Bigg] \\
&\le \left(b_{21}\, \phi_1 + \sum_{j=2}^{r} b_{2j}\right) L\, \|\mathbf{Y}_0\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right) \\
&\triangleq \phi_2\, L\, \|\mathbf{Y}_0\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right);
\end{aligned} \tag{3.8}$$

$$\cdots \qquad \cdots$$

$$\begin{aligned}
\left\|\mathbf{y}_r^{(1)}(t) - \mathbf{y}_r^{(0)}(t)\right\| &\le (M_r + c_r\, \alpha_r) \int_{t_0}^{t} \exp\left(-\int_{s}^{t} \lambda_r(\xi)\, d\xi\right) \\
&\quad \times \Bigg[\sum_{j=1}^{r-1} \left\|\mathbf{y}_j^{(1)}(s) - \mathbf{y}_j^{(0)}(s)\right\| (f_{rj}(s)\, d|u_j|(s) + g_{rj}(s)\, d|u_r|(s)) \\
&\quad + \left\|\mathbf{y}_r^{(0)}(s)\right\| (f_{rr}(s) + g_{rr}(s))\, d|u_r|(s)\Bigg] \\
&\le \left(\sum_{j=1}^{r-1} b_{rj}\, \phi_j + b_{rr}\right) L\, \|\mathbf{Y}_0\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right) \\
&\triangleq \phi_r\, L\, \|\mathbf{Y}_0\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right),
\end{aligned} \tag{3.9}$$

where $\phi_1 = \sum_{j=1}^{r} b_{1j}$, $\phi_2 = b_{21}\, \phi_1 + \sum_{j=2}^{r} b_{2j}, \ldots, \phi_r = \sum_{j=1}^{r-1} b_{rj}\, \phi_j + b_{rr}$, which are equivalent to

$$\begin{aligned}
&\mathrm{Col}(\phi_1, \ldots, \phi_r) = \mathbf{B}_1\, \mathrm{Col}(\phi_1, \ldots, \phi_r) + \mathbf{B}_2\, \mathrm{Col}(1, \ldots, 1), \qquad \text{or} \\
&\mathrm{Col}(\phi_1, \ldots, \phi_r) = (\mathbf{E} - \mathbf{B}_1)^{-1}\, \mathbf{B}_2\, \mathrm{Col}(1, \ldots, 1).
\end{aligned}$$

Summing up the inequalities (3.7)–(3.9), we have

$$\begin{aligned}
&\mathrm{Col}\left(\left\|\mathbf{y}_1^{(1)}(t) - \mathbf{y}_1^{(0)}(t)\right\|, \ldots, \left\|\mathbf{y}_r^{(1)}(t) - \mathbf{y}_r^{(0)}(t)\right\|\right) \\
&\quad \le \mathrm{Col}(\phi_1, \ldots, \phi_r)\, L\, \|\mathbf{Y}_0\| \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right) \\
&\quad = L\, \|\mathbf{Y}_0\|\, (\mathbf{E} - \mathbf{B}_1)^{-1}\, \mathbf{B}_2\, \mathrm{Col}(1, \ldots, 1) \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right) \\
&\quad = L\, \mathbf{D}\, \|\mathbf{Y}_0\|\, \mathrm{Col}(1, \ldots, 1) \exp\left(-\int_{t_0}^{t} \lambda(\xi)\, d\xi\right).
\end{aligned} \tag{3.10}$$

Now, we assume that

$$\begin{aligned}
&\mathrm{Col}\left(\left\|\mathbf{y}_1^{(m)}(t)-\mathbf{y}_1^{(m-1)}(t)\right\|,\dots,\left\|\mathbf{y}_r^{(m)}(t)-\mathbf{y}_r^{(m-1)}(t)\right\|\right)\\
&\quad\le \mathrm{Col}\left(\phi_1^{(m)},\dots,\phi_r^{(m)}\right) L\,\|\mathbf{Y}_0\| \exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right)\\
&\quad= L\,\mathbf{D}^m\,\|\mathbf{Y}_0\|\,\mathrm{Col}(1,\dots,1)\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right),
\end{aligned} \tag{3.11}$$

where $\phi_j^{(1)}=\phi_j$, $j=1,\dots,r$. Our objective is to show that the inequality (3.11) holds for the integer $m+1$. In fact,

$$\begin{aligned}
\left\|\mathbf{y}_1^{(m+1)}(t)-\mathbf{y}_1^{(m)}(t)\right\| &\le (M_1+c_1\,\alpha_1)\int_{t_0}^{t}\exp\left(-\int_{s}^{t}\lambda_1(\xi)\,d\xi\right)\\
&\quad\times\sum_{j=1}^{r}\left\|\mathbf{y}_j^{(m)}(s)-\mathbf{y}_j^{(m-1)}(s)\right\|\left[f_{1j}(s)\,d|u_j|(s)+g_{1j}(s)\,d|u_1|(s)\right]\\
&\le\sum_{j=1}^{r}b_{1j}\,\phi_j^{(m)}\,L\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right)\\
&\triangleq\phi_1^{(m+1)}\,L\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right);
\end{aligned}$$

$$\begin{aligned}
\left\|\mathbf{y}_2^{(m+1)}(t)-\mathbf{y}_2^{(m)}(t)\right\| &\le (M_2+c_2\,\alpha_2)\int_{t_0}^{t}\exp\left(-\int_{s}^{t}\lambda_2(\xi)\,d\xi\right)\\
&\quad\times\Bigg[\left\|\mathbf{y}_1^{(m+1)}(s)-\mathbf{y}_1^{(m)}(s)\right\|\left(f_{21}(s)\,d|u_1|(s)+g_{21}(s)\,d|u_2|(s)\right)\\
&\quad+\sum_{j=2}^{r}\left\|\mathbf{y}_j^{(m)}(s)-\mathbf{y}_j^{(m-1)}(s)\right\|\left(f_{2j}(s)\,d|u_j|(s)+g_{2j}(s)\,d|u_2|(s)\right)\Bigg]\\
&\le\left(b_{21}\,\phi_1^{(m+1)}+\sum_{j=2}^{r}b_{2j}\,\phi_j^{(m)}\right)L\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right)\\
&\triangleq\phi_2^{(m+1)}\,L\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right);
\end{aligned}$$

$$\cdots\quad\cdots\quad\cdots$$

$$\begin{aligned}
\left\|\mathbf{y}_r^{(m+1)}(t)-\mathbf{y}_r^{(m)}(t)\right\| &\le (M_r+c_r\,\alpha_r)\int_{t_0}^{t}\exp\left(-\int_{s}^{t}\lambda_r(\xi)\,d\xi\right)\\
&\quad\times\Bigg[\sum_{j=1}^{r-1}\left\|\mathbf{y}_j^{(m+1)}(s)-\mathbf{y}_j^{(m)}(s)\right\|\left(f_{rj}(s)\,d|u_j|(s)+g_{rj}(s)\,d|u_r|(s)\right)\\
&\quad+\left\|\mathbf{y}_r^{(m)}(s)-\mathbf{y}_r^{(m-1)}(s)\right\|\left(f_{rr}(s)+g_{rr}(s)\right)d|u_r|(s)\Bigg]\\
&\le\left(\sum_{j=1}^{r-1}b_{rj}\,\phi_j^{(m+1)}+b_{rr}\,\phi_r^{(m)}\right)L\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right)\\
&\triangleq\phi_r^{(m+1)}\,L\,\|\mathbf{Y}_0\|\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right),
\end{aligned}$$

where

$$\phi_1^{(m+1)} = \sum_{j=1}^{r} b_{1j}\,\phi_j^{(m)},$$

$$\phi_2^{(m+1)} = b_{21}\,\phi_1^{(m+1)} + \sum_{j=2}^{r} b_{2j}\,\phi_j^{(m)}, \dots, \phi_r^{(m+1)} = \sum_{j=1}^{r-1} b_{rj}\,\phi_j^{(m+1)} + b_{rr}\,\phi_r^{(m)},$$

which are equivalent to

$$\mathrm{Col}\left(\phi_1^{(m+1)},\dots,\phi_r^{(m+1)}\right) = \mathbf{B}_1\ \mathrm{Col}\left(\phi_1^{(m+1)},\dots,\phi_r^{(m+1)}\right) + \mathbf{B}_2\ \mathrm{Col}\left(\phi_1^{(m)},\dots,\phi_r^{(m)}\right), \qquad \text{or}$$

$$\begin{aligned}\mathrm{Col}\left(\phi_1^{(m+1)},\dots,\phi_r^{(m+1)}\right) &= (\mathbf{E}-\mathbf{B}_1)^{-1}\,\mathbf{B}_2\ \mathrm{Col}\left(\phi_1^{(m)},\dots,\phi_r^{(m)}\right)\\ &= \mathbf{D}\ \mathrm{Col}\left(\phi_1^{(m)},\dots,\phi_r^{(m)}\right).\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}&\mathrm{Col}\left(\left\|\mathbf{y}_1^{(m+1)}(t)-\mathbf{y}_1^{(m)}(t)\right\|,\dots,\left\|\mathbf{y}_r^{(m+1)}(t)-\mathbf{y}_r^{(m)}(t)\right\|\right)\\ &\quad\le L\,\|\mathbf{Y}_0\|\ \mathrm{Col}\left(\phi_1^{(m+1)},\dots,\phi_r^{(m+1)}\right)\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right)\\ &\quad= L\,\|\mathbf{Y}_0\|\,\mathbf{D}^{m+1}\ \mathrm{Col}(1,\dots,1)\ \exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right).\end{aligned}$$

By mathematical induction, we have claim that for all natural numbers, the estimate (3.11) holds, and

$$\begin{aligned}\mathrm{Col}\left(\left\|\mathbf{y}_1^{(m)}(t)\right\|,\dots,\left\|\mathbf{y}_r^{(m)}(t)\right\|\right) &\le \mathrm{Col}\left(\left\|\mathbf{y}_1^{(m)}(t)-\mathbf{y}_1^{(m-1)}(t)\right\|,\dots,\left\|\mathbf{y}_r^{(m)}(t)-\mathbf{y}_r^{(m-1)}(t)\right\|\right)\\ &\quad+\dots+\mathrm{Col}\left(\left\|\mathbf{y}_1^{(1)}(t)-\mathbf{y}_1^{(0)}(t)\right\|,\dots,\left\|\mathbf{y}_r^{(1)}(t)-\mathbf{y}_r^{(0)}(t)\right\|\right)\\ &\quad+\mathrm{Col}\left(\left\|\mathbf{y}_1^{(0)}(t)\right\|,\dots,\left\|\mathbf{y}_r^{(0)}(t)\right\|\right)\\ &\le (\mathbf{D}^m+\dots+\mathbf{D}+\mathbf{E})\,L\,\|\mathbf{Y}_0\|\ \mathrm{Col}(1,\dots,1)\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right)\\ &\le (\mathbf{E}-\mathbf{D})^{-1}\,L\,\|\mathbf{Y}_0\|\ \mathrm{Col}(1,\dots,1)\ \exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right).\end{aligned} \tag{3.12}$$

From (3.12), it is clear that, if

$$\|\mathbf{Y}_0\|H/\left[c\,b^{r-1}\ \left\|(\mathbf{E}-\mathbf{D})^{-1}\right\|\ \|\mathrm{Col}(1,\dots,1)\|\ \sup_{t\ge t_0}\exp\left(-\int_{t_0}^{t}\lambda(\xi)\,d\xi\right)\right] \triangleq H_1, \tag{3.13}$$

then $\left\|\mathbf{Y}^{(m)}(t)\right\| < H$, which implies that a sequence $\left\{\mathbf{Y}^{(m)}(t)\right\}$ exists. Note that $\rho(\mathbf{D}) < 1$, hence, for any interval $[t_{k-1}, t_k)$, $(k \in \mathbb{N})$, the series $\sum_{m=1}^{\infty}\mathbf{D}^m\ \mathrm{Col}(1,\dots,1)$ is convergent. Therefore,

$$\begin{aligned}&\sum_{m=1}^{\infty}\mathrm{Col}\left(\left\|\mathbf{y}_1^{(m)}(t)-\mathbf{y}_1^{(m-1)}(t)\right\|,\dots,\left\|\mathbf{y}_r^{(m)}(t)-\mathbf{y}_r^{(m-1)}(t)\right\|\right)\\ &\qquad+\mathrm{Col}\left(\left\|\mathbf{y}_1^{(0)}(t)\right\|,\dots,\left\|\mathbf{y}_r^{(0)}(t)\right\|\right), \qquad \text{and}\\ &\sum_{m=1}^{\infty}\mathrm{Col}\left(\mathbf{y}_1^{(m)}(t)-\mathbf{y}_1^{(m-1)}(t),\dots,\mathbf{y}_r^{(m)}(t)-\mathbf{y}_r^{(m-1)}(t)\right)\\ &\qquad+\mathrm{Col}\left(\mathbf{y}_1^{(0)}(t),\dots,\mathbf{y}_r^{(0)}(t)\right)\end{aligned}$$

are uniformly convergent series, which implies that

$$\left\{\mathrm{Col}\left(\mathbf{y}_1^{(m)}(t),\dots,\mathbf{y}_r^{(m)}(t)\right)\right\},$$

is uniformly convergent on $[t_{k-1},t_k)$. Setting

$$\lim_{m\to\infty}\mathrm{Col}\left(\mathbf{y}_1^{(m)}(t),\dots,\mathbf{y}_r^{(m)}(t)\right)=\mathrm{Col}\left(\mathbf{y}_1(t),\dots,\mathbf{y}_r(t)\right),$$

and in view of that $\mathbf{F}_{ij}(t,\mathbf{y}_j)$ and $\mathbf{G}_i(t,\mathbf{Y})$ are continuous with respect to $\mathbf{y}_j$ and $\mathbf{Y}$, respectively, we have

$$\lim_{m\to\infty}\mathbf{F}_{ij}\left(t,\mathbf{y}_j^{(m)}(t)\right)=\mathbf{F}_{ij}(t,\mathbf{y}_j(t)),\quad \lim_{m\to\infty}\mathbf{G}_i(t,\mathbf{Y}^{(m)}(t))=\mathbf{G}_i(t,\mathbf{Y}(t)).$$

Noticing Conditions (H$_1$), (H$_3$), and that $\left\|\mathbf{Y}^{(m)}(t)\right\|<H$, by the Lebesgue-Stieltjes dominated convergence theorem, we can see that $\mathbf{Y}(t)=\mathrm{Col}\left(\mathbf{y}_1(t),\dots,\mathbf{y}_r(t)\right)$, $t\in[t_{k-1},t_k)$ is the solution of (2.1), which is a right-continuous and bounded variation. Noticing that all estimates we have obtained are independent of the integer k, and considering inequality (3.12), we get.

$$\mathrm{Col}\left(\|\mathbf{y}_1(t)\|,\dots,\|\mathbf{y}_r(t)\|\right)\le(\mathbf{E}-\mathbf{D})^{-1}cb^{r-1}\,\mathrm{Col}(1,\dots,1)\exp\left(-\int_{t_0}^{t}\lambda(s)\,ds\right),\qquad t\ge t_0. \tag{3.14}$$

From this estimate, we can see that Conclusion (I) holds. In view of (3.13), Conclusion (II) also holds. This completes the proof. ∎

COROLLARY 3.1. *Replace $\rho(\mathbf{D})<1$ with $\|\mathbf{D}\|<1$ in Theorem 3.1 and suppose that the other assumptions of Theorem 3.1 hold, where $\|\mathbf{D}\|$ is given by one of the following formulae*

$$\|\mathbf{D}\|=\max_{1\le j\le r}\sum_{i=1}^{r}d_{ij};$$

$$\|\mathbf{D}\|_2=\left(\lambda_{\max}\left(\mathbf{D}^T\mathbf{D}\right)\right)^{1/2};$$

$$\|\mathbf{D}\|_\infty=\max_{1\le i\le r}\sum_{j=1}^{r}d_{ij};$$

$$\|\mathbf{D}\|_E=\left(\sum_{i,j=1}^{r}d_{ij}^2\right)^{1/2}.$$

Then:

(I) *Conclusion* (I) *of Theorem 3.1 holds;*

(II) *the attractive region of the system* (2.1) *is given by*

$$\|\mathbf{Y}_0\|\le H_1^*\equiv H/\left[cb^{r-1}\left(\|\mathbf{E}\|+\frac{\|\mathbf{D}\|}{1-\|\mathbf{D}\|}\right)\|\mathrm{Col}(1,\dots,1)\|\sup_{t\ge t_0}\exp\left(-\int_{t_0}^{t}\lambda(s)\,ds\right)\right].$$

PROOF. Since $\rho(\mathbf{D})\le\|\mathbf{D}\|<1$, that Conclusion (I) holds is clear. Note that

$$\|(\mathbf{E}-\mathbf{D})^{-1}\|\le\sum_{m=0}^{\infty}\|\mathbf{D}^m\|\le\|\mathbf{E}\|+\sum_{m=1}^{\infty}\|\mathbf{D}\|^m=\|\mathbf{E}\|+\frac{\|\mathbf{D}\|}{(1-\|\mathbf{D}\|)},$$

which implies that Conclusion (II) holds; the proof of the corollary is complete. ∎

THEOREM 3.2. *Suppose that:*

(i) *Assumption* (H$_1$) *holds, and* $f_{ij}(t)\equiv f_{ij}=\text{const.}$, $g_{ij}(t)\equiv g_{ij}=\text{const.}$;

(ii) *There exist constants $M_i>0$ and $\lambda_i>0$ ($i=1,\dots,r$) such that*

$$\|P_i(t,t_0)\|\le M_i\exp(-\lambda_i(t-t_0)),\qquad t\ge t_0;$$

(iii) $\rho(\mathbf{D}^*)<1$, *where* $\mathbf{D}^*=(\mathbf{E}-\mathbf{B}_1^*)^{-1}\mathbf{B}_2^*$, $\mathbf{B}^*=\left((M_i+\alpha_i c_i)\left(f_{ij}a_j+g_{ij}a_i\right)\right)_{r\times r}$, $a_j=\sum_{k=1}^{\infty}|\mathbf{a}_{jk}|<\infty$, α_i, *and c_i are defined in* (H$_3$).

Then:

(I) *the trivial solution of the system* (2.1) *is exponentially stable;*
(II) *the attractive region of the system* (2.1) *is*

$$\|\mathbf{Y}_0\| < H_1^* \equiv H/\left[c\, b^{r-1} \left\|(\mathbf{E}-\mathbf{D}^*)^{-1}\right\| \, \|\operatorname{Col}(1,\dots,1)\|\right]. \tag{3.15}$$

PROOF. Choosing a constant λ such that $0 < \lambda < \lambda_i$ $(i = 1,\dots,r)$, hence,

$$\begin{aligned}
&\|P_i(t,t_0)\| \le M_i \exp\left(-\int_{t_0}^{t} \lambda_i \, d\xi\right) \le M_i \exp\left(-\lambda(t-t_0)\right);\\
&\qquad \int_{t_0}^{t} (M_i + \alpha_i c_i) \exp\left(-\int_{s}^{t} (\lambda_i - \lambda)\, d\xi\right) \times \left[f_{ij}\, d|u_j|(s) + g_{ij}\, d|u_i|(s)\right]\\
&\qquad \le (M_i + \alpha_i c_i)\left[f_{ij}\, a_j + g_{ij}\, a_i\right] = b_{ij},
\end{aligned}$$

which shows that Assumptions (H_2) and (H_3) hold. It is similar to the inference of Theorem 3.1; we get

$$0 \le \operatorname{Col}\left(\|\mathbf{y}_1(t)\|,\dots,\|\mathbf{y}_r(t)\|\right) \le (\mathbf{E}-\mathbf{D}^*)^{-1} c\, b^{r-1} \operatorname{Col}(1,\dots,1) \exp\left(-\lambda(t-t_0)\right), \qquad t \ge t_0,$$

which implies that Conclusion (I) holds. With similar discussion to (3.13), we have (3.15), which shows that Conclusion (II) holds and the proof is complete. ∎

4. CONCLUSION

Stability problems of nonlinear large scale measure systems with impulsive solutions are presented and taken into account in this paper. The lumped Gauss-Seidel iteration method we used avoided the difficulties of constructing a Lyapunov function. The criteria of stability obtained are simple and practical, although the proofs of the theorems seem to be complicated.

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